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THE SPECTRUM OF A NEUTRON
TRANSPORT OPERATOR

by

Roger N. van Norton

June 1, 1960

Institute of Mathematical Sciences

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THE SPECTRUM OF A NEUTRON TRANSPORT OPERATOR

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ABSTRACT

The spectrum of the mono-energetic neutron transport operator for a homogeneous sphere where the neutrons are scattered with spherical symmetry in the laboratory system is considered. The operator may be written:

$$Af \equiv -\frac{\partial f(x,y)}{\partial x} - \sigma f(x,y) + \frac{c}{2r}\,\phi(r)$$

$$\phi(r) \equiv \int_{-r}^{r} f(z,\sqrt{r^2-z^2})\,dz$$

in the region $r^2 = x^2 + y^2 \leq a^2$, $0 \leq y$ with the boundary condition $f(-\sqrt{a^2-y^2},y) = 0$ and with σ and c real, positive constants.

The characteristic value problem $Af = \lambda f$ is to be analyzed. An associated integral transformation

$$L_\lambda \phi \equiv \int_0^a \phi(s)\,ds \int_{|r-s|}^{|r+s|} e^{-\lambda w}\,\frac{dw}{w}$$

arises in a natural way. For real λ the operator L_λ is hermitian. The characteristic value problem

$$\rho\phi = L_\lambda \phi$$

will, with the relation

$$f(x,y) = \frac{c}{2} \int_{-\sqrt{a^2-y^2}}^{x} \phi(\sqrt{z^2+y^2}) \, \frac{e^{-\lambda(x-z)}}{\sqrt{z^2+y^2}} \, dz$$

yield solutions of the problem when $\rho = 2/c$.

The real spectrum of A is analyzed by investigation of the integral transformation L_λ for real λ . In this case it is shown that its characteristic values $\rho_i(\lambda)$ $i = 1,2,\ldots$ are positive and depend continuously on λ . Comparison transformations show that

$$\lim_{\lambda \longrightarrow \infty} \rho_i(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda \longrightarrow -\infty} \rho_i(\lambda) = \infty .$$

This proves the existence of an infinite number of real characteristic values of A . These characteristic values are shown to have index one.

The proof is based on an extension of methods used by Lehner and Wing [3].

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1. Introduction

The neutron density at time t in a body may be described by a non-negative function $\psi(\underline{x},\underline{v},t)$ where $\underline{x} = (x_1,x_2,x_3)$ and $\underline{v} = (v_1,v_2,v_3)$. This function satisfies the linear Boltzmann equation

$$\frac{\partial\psi}{\partial t} = -\underline{v} \cdot \operatorname{grad}_x \psi - \sigma(\underline{x},v)\,\psi + \int_V K(\underline{x},\underline{v},\underline{v}')\psi(\underline{x},\underline{v}',t)d\underline{v}' \qquad (1.1)$$

for x in a domain X, v in a domain V and positive t. Usually X is assumed to be convex; in this case a standard boundary condition is $\psi(\underline{x},\underline{v},t) = 0$ for $\underline{x}$ on the boundary of X and $\underline{v}.\underline{n} < 0$, where $\underline{n}$ is the outward normal to the domain X (i.e. no neutrons enter X from outside X). The domain V is bounded. The non-negative function $\sigma(\underline{x},v)$ is the macroscopic removal cross-section multiplied by v for neutrons at $\underline{x}$ with speed v. The non-negative kernel $K(\underline{x},\underline{v},\underline{v}')$ is a function or distribution and the integral transformation describes the production of neutrons by fission or scattering at $\underline{x}$ with velocity $\underline{v}$.

The initial value problem is to find $\psi(\underline{x},\underline{v},t)$ for $t > 0$ which assumes an initial density $\psi(\underline{x},\underline{v},0)$ as $t \longrightarrow 0+$.

K. Jorgens has shown under rather general conditions that this is a well-posed initial value problem for $X \times V$ bounded. The existence of a strongly continuous semi-group operator, $E(t)$, which solves the initial value problem is demonstrated. In some cases, additional assumptions assure that $E(t)$ is compact after some finite time which depends on the domain $X \times V$, the character of the scattering operator, and the underlying Banach space on which the problem is posed. In such a case the spectrum of A is discrete, confined to a left half plane and finite in every finite vertical strip ([1] Th. 5.1). Furthermore, an asymptotic expansion of $E(t)$, the semi-group operator, in terms of the projection operators associated with the generalized characteristic functions of A exists ([1] Th. 5.2).

In an earlier series of papers Lehner and Wing ([2],[3],[4]) have completely analyzed the spectrum of the transport operator for a simple case, the infinite slab of finite width where the neutrons have only one speed and are scattered with spherical symmetry in the laboratory system. This problem may be written

$$\frac{1}{v}\frac{\partial \psi}{\partial t}(z,\mu,t) = -\mu\frac{\partial \psi}{\partial x} - \sigma\psi + \frac{c}{2}\int_{-1}^{1} \psi(z,\mu',t)\,d\mu' \qquad (1.2)$$

$$\psi(a,\mu,t) = 0 \qquad \text{for } \mu < 0$$

$$\psi(-a,\mu,t) = 0 \qquad \text{for } \mu > 0$$

$$\psi(z,\mu,0)$$

is given on

$$X : \ |z| \leq a \ ; \ |\mu| \leq 1$$

$$V. : \ v \ , \text{ a positive constant}$$

$$\psi \in L_2(|z| \leq a \ ; \ |\mu| \leq 1)$$

Here σ and c are positive constants. The domain $X \times V$ is unbounded and Jörgens' conclusions do not apply. The point spectrum of the operator is shown to be finite but not empty, the continuous spectrum fills a left half plane, and the residual spectrum is empty. Lehner also partially analyzed the same problem where X is a finite sphere ([4]). Here he determines in accordance with Jörgens that the operator has a discrete spectrum with $-\infty$ the only possible accumulation point.

2. The Mono-energetic Neutron Transport Operator for the Homogeneous Sphere.

With the following simplifying specializations

(i) X is the domain $r^2 = |\underline{x}|^2 \leq a^2$,

(ii) V is the domain $|\underline{v}| = \text{const.} > 0$,

(iii) $\sigma(\underline{x}, v)$ is a positive constant,

(iv) Scattering is spherically symmetric in the laboratory system,

the problem (1.1) becomes

$$\frac{1}{v}\frac{\partial \psi(r,\mu,t)}{\partial t} = -\mu\frac{\partial \psi}{\partial r} - \frac{1-\mu^2}{r}\frac{\partial \psi}{\partial \mu} - \sigma\psi + \tag{2.1}$$

$$+ \frac{c}{2}\int_{-1}^{+1} \psi(r,\mu',t)\, d\mu'$$

with boundary condition

$$\psi(a,\mu,t) = 0 \qquad \text{for } \mu < 0$$

and the initial condition

$$\psi(r,\mu,0) \quad \text{prescribed on } X \times V .$$

Here R is the distance from the center of the sphere, μ is the cosine of the angle the neutron beam makes with the radius vector, and σ and c are real, positive constants.

It will be convenient to perform a change of

variables before rendering the problem more specific. Let:

$$x = r\,\mu$$
$$y = r\sqrt{1-\mu^2}$$
$$\psi(r,\mu,t) = f(x,y,t) \quad ;$$

then

$$\frac{\partial f(x,y,t)}{\partial t} = Af \tag{2.2}$$

$$Af \equiv -\frac{\partial f}{\partial x} - \sigma f + \frac{c}{2r}\int_{-r}^{+r} f(x_1, \sqrt{r^2-x_1^2}, t)\, dx_1$$

with the initial data $f(x,y,0)$ prescribed. The operator A is defined over a Hilbert space H of square integrable functions with the inner product

$$(f,g) = 4\pi \int_0^a y\, dy \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x,y)\, \overline{g(x,y)}\, dx \ .$$

At first glance this inner product may seem unusual, but it is merely the transformation of the natural inner product for the operator in terms of the original independent variables.

$$(\psi_1,\psi_2) = 4\pi \int_0^a r^2\, dr \int_{-1}^{+1} d\mu\, \psi_1\, \overline{\psi_2}$$

The domain, $D(A)$, of the operator in (2.2) consists of

those functions f which satisfy:

(i) $f(x,y)$ is defined for $r^2=x^2+y^2 \leq a^2$ and $y \geq 0$,

(ii) $f \in H$,

(iii) $Af \in H$,

(iv) $f(-\sqrt{a^2-y^2},y) = 0$

In particular, Af is absolutely continuous in x for all y.

This is a well-posed initial value problem; the operator has a discrete spectrum with $-\infty$ as the only possible accumulation point ([1] and [4]).

If $f(x,y)$ is in $D(A)$ and

$$(\frac{a}{v} + \sigma) f(x,y) = -\frac{\partial f}{\partial x} + \frac{c}{2r}\int_{-r}^{r} f(z,\sqrt{r^2-z^2})\, dz$$

then $e^{at/v} f(x,y)$ satisfies (2.2) with initial data given to be $f(x,y)$ and f is a characteristic function of A with characteristic value a/v.

Abbreviate $\lambda = a/v + \sigma$ and define

$$\phi(r) = \int_{-r}^{r} f(z,\sqrt{r^2-z^2})dz \,. \qquad (2.3)$$

(Notice that ϕ is an odd function.) Rewrite

$$\lambda f(x,y) + \frac{\partial f(x,y)}{\partial x} = \frac{c}{2r} \phi(r) .$$

Since $f \in D(A)$

$$f(x,y) = \frac{c}{2} e^{-\lambda x} \int_{-\sqrt{a^2-y^2}}^{x} \phi(\sqrt{x_1^2+y^2}) \frac{e^{\lambda x_1}}{\sqrt{x_1^2+y^2}} dx_1$$

and from (2.3)

$$\phi(r) = \int_{-r}^{r} f(x_2, \sqrt{r^2-x_2^2}) \, dx_2$$

$$= \frac{c}{2} \int_{-r}^{r} e^{-\lambda x_2} dx_2 \int_{-\sqrt{a^2-r^2+x_2^2}}^{x_2} \phi(\sqrt{x_1^2+r^2-x_2^2}) \frac{e^{\lambda x_1}}{\sqrt{x_1^2+r^2-x_2^2}} dx_1$$

The change of dummy variables

$$s^2 = x_1^2 + r^2 - x_2^2$$

$$w = x_2 - x_1 \qquad \text{yields}$$

$$\phi(r) = \frac{c}{2} L_\lambda \phi \qquad \text{where} \qquad (2.4)$$

$$L_\lambda \phi \equiv \int_0^a \phi(s) \, ds \int_{|r-s|}^{|r+s|} e^{-\lambda w} \frac{dw}{w}$$

This integral equation will be the focus of attention.

Theorem 2.1. The necessary and sufficient condition for $f(x,y)$ to be a characteristic function of the transport

equation is that its associated ϕ be a characteristic function of the integral equation $\rho\phi = L_\lambda \phi$ with characteristic value $\rho = 2/c$.

Proof. The derivation of the integral equation has proved sufficiency. The necessity follows by reversing the computation. That $\phi(r)$ as defined in (2.3) is in $L_2(0,a)$ is a consequence of $f \in D(A)$ and

$$(\phi,\phi) = \int_0^a dr \left| \int_{-r}^{r} f(z, \sqrt{r^2-z^2})\, dz \right|^2$$

$$\leq \int_0^a dr\, 2r \int_{-r}^{r} \left| f(z, \sqrt{r^2-z^2}) \right|^2 dz$$

$$= 2 \int_0^a y\, dy \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} |f(x,y)|^2\, dx$$

$$= \frac{1}{2\pi} (f,f) < \infty$$

In what follows it may be assumed that ϕ has norm one, since no explicit use will be made of the norm of the characteristic functions of the transport operator except that it is not zero.

The knowledge of solutions $\phi(r;\lambda)$ of (2.4) will provide information about the spectrum of the transport operator. The integral equation $\rho\phi = L_\lambda \phi$ resembles the usual characteristic value problem. However, ρ is a

positive constant dependent on the nuclear properties of the materials in the sphere. Therefore, only characteristic functions $\phi(r;\lambda)$ with positive characteristic values, ρ, are to be considered and only those with characteristic value $2/c$ will be the characteristic functions sought. A list of properties of L_λ follows.

Theorem 2.2. The integral transformation, L_λ, defined for all complex λ with domain $D(L_\lambda)$ taken to be the Hilbert space $L_2(0,a)$ with the ordinary inner product, has the following properties.

(i) Its kernel is symmetric but not hermitian (unless λ is real).

(ii) It is compact.

(iii) It is non-degenerate for real λ.

Proof. Property (i) is obvious. Notice that for λ real the kernel is a positive function. Therefore the transformation maps non-trivial non-negative functions into positive functions. Coupled with the compactness of the operator this will imply that the dominant characteristic value of L_λ is simple for real λ ([7]).

The kernel of L_λ (2.4) may be written as $|r-s|^{-\varepsilon} B(r,s)$ where ε is an arbitrarily small positive number and $B(r,s)$ is a bounded, continuous function of r and s. Therefore, the transformation is compact ([6], p.326).

If L_λ were degenerate for real λ, its kernel could be written as a finite sum $\sum \rho_i \phi_i(r) \overline{\phi_i(s)}$ where the ϕ_i would be the normalized characteristic functions of

the operator. Therefore the trace of the kernel would exist and be finite. However, this is not the case.

In short L_λ is, for real λ, an integral transformation of Hilbert-Schmidt type.

If ϕ is extended as odd to the interval $|r| \leq a$, then for $\beta \equiv \mathrm{Re}(\lambda) > 0$, (2.4) may be rewritten

$$\rho\,\phi(r) = \int_0^a \phi(s)\,ds \int_{|r-s|}^{\infty} e^{-\lambda w}\,\frac{dw}{w} - \int_0^a \phi(s)\,ds \int_{|r+s|}^{\infty} e^{-\lambda w}\,\frac{dw}{w}$$

$$= \int_0^a \phi(s)\,E(\lambda|r-s|)\,ds - \int_0^{-a} \phi(s)\,E(\lambda|r-s|)\,ds$$

$$\rho\,\phi(r) = \int_{-a}^{a} \phi(s)\,E(\lambda|r-s|)\,ds \qquad \mathrm{Re}(\lambda) > 0 \qquad (2.5)$$

which is the integral equation associated with problem (1.2) for the infinite slab of width $2a$ ([2]).

Lehner and Wing ([2],[3]) showed that (2.5) has solutions with positive real ρ only if $\tau \equiv \mathrm{Im}(\lambda) = 0$ and $\beta > 0$. In the case $\tau = 0$ and $\beta > 0$

$$\int_{-a}^{a} \phi(s)\,E(\beta|r-s|)\,ds$$

is a positive definite operator with positive characteristic values $\rho_1 > \rho_2 \geq \cdots$ and for each i, ρ_i is a continuous decreasing function of β which tends to zero as β tends to infinity. Also, ρ_1 approaches

infinity as β approaches 0 from above ([2]). This is summarized in Figure 1. These characteristic functions may be taken to be odd or even. In particular, the dominant eigenvalue is simple and corresponds to a characteristic function of one sign, hence even ([7], Th. 6.3). Deleting the even functions in order to obtain solutions for the sphere might lead to a graph like Figure 2. There will be at most a finite set of values of β, and possibly none, which yield characteristic value $2/c$. The corresponding characteristic functions and values of β for this integral equation problem will yield characteristic functions and characteristic values for the transport equation.

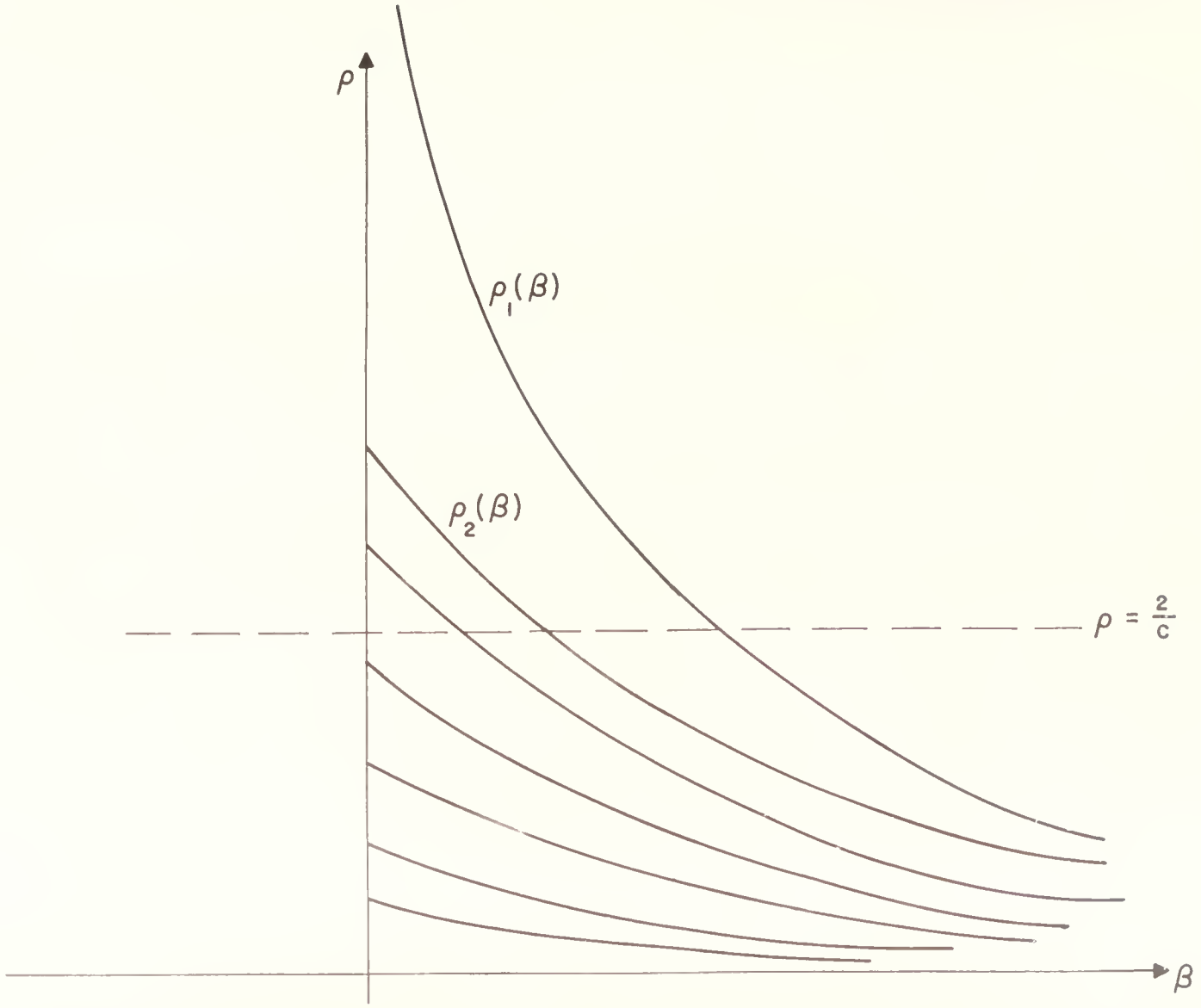

Figure 1

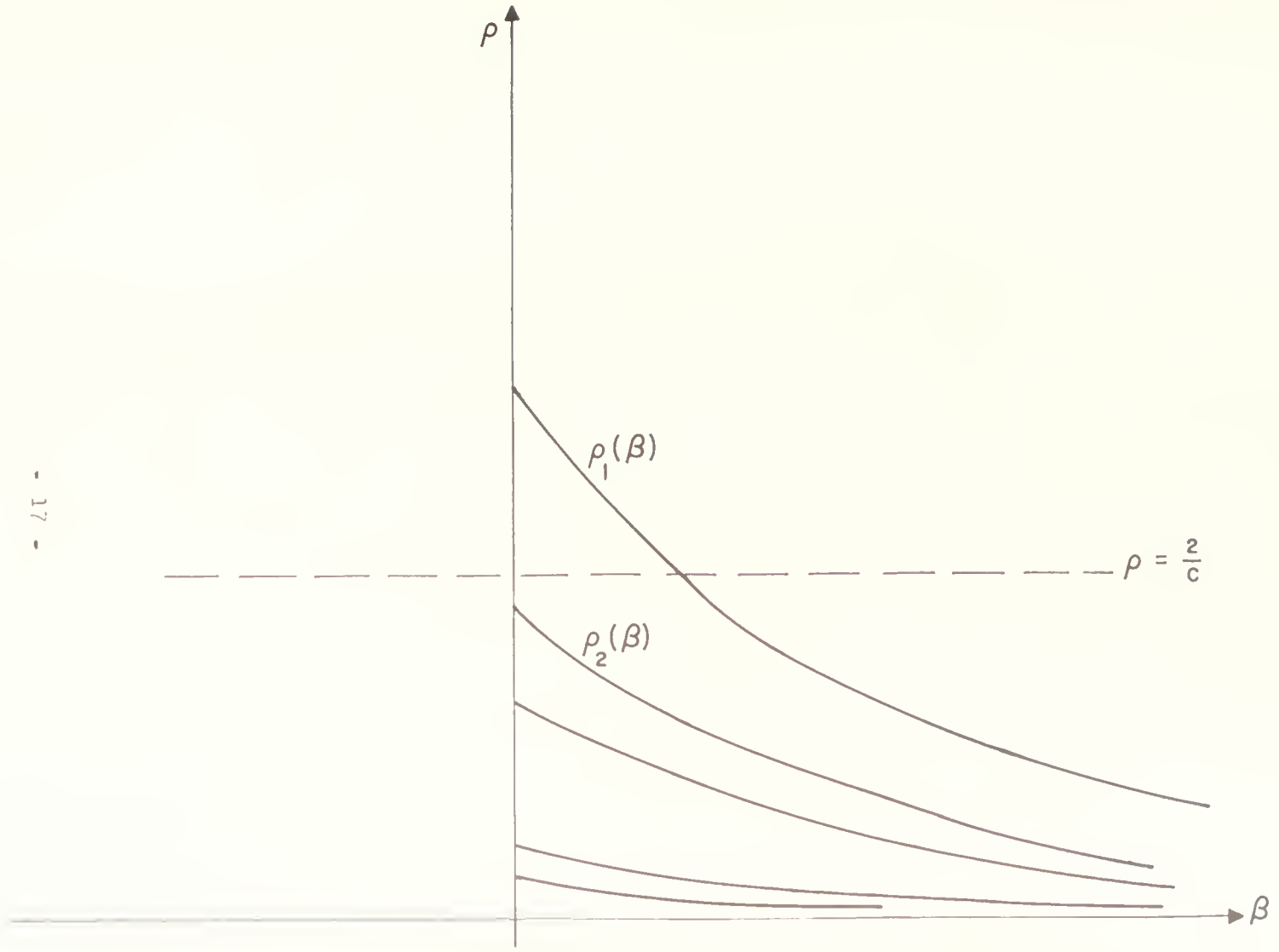

Figure 2

3. The Real Spectrum of the Operator

The integro-differential operator is:

$$Af = -\frac{\partial f(x,y)}{\partial x} - \sigma f(x,y) + \frac{c}{2r}\phi(r) , \qquad (3.1)$$

$$\phi(r) = \int_{-r}^{r} f(z, \sqrt{r^2-z^2})\, dz$$

on

$$X : r^2 = x^2 + y^2 \leq a^2 \qquad y \geq 0 .$$

The domain $D(A)$ consists of functions f in a Hilbert space H of functions square integrable on X such that

$$f \in H ; \quad Af \in H ; \quad f(-\sqrt{a^2-y^2}, y) = 0$$

where the inner product is defined to be

$$(f,g) = 4\pi \int_0^a y\, dy \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x,y)\, \overline{g(x,y)}\, dx .$$

The adjoint operator acts on the same Hilbert space and is

$$A^* f = +\frac{\partial f(x,y)}{\partial x} - \sigma f(x,y) + \frac{c}{2r}\phi(r) . \qquad (3.2)$$

Its domain $D(A^*)$ consists of functions f such that

$$f \in H ; \quad A^* f \in H ; \quad f(+\sqrt{a^2-y^2}, y) = 0 .$$

Since $\sigma(A)$, the spectrum of A, is discrete,

$\sigma(A) = \overline{\sigma(A^*)}$ and the spectrum of A^* is discrete ([8], p. 143). Notice further that if $\lambda \in \sigma(A)$ with characteristic function $f(x,y)$, there naturally corresponds to it a $\overline{\lambda} \in \sigma(A)$ with characteristic function $\overline{f(x,y)}$. Thus, the spectrum of A is symmetric with respect to the real axis of the complex plane.

If $f(x,y)$ is a characteristic function with characteristic value λ, then

$$\frac{2}{c}\,\phi(r) = \int_0^a \phi(s)\,ds \int_{|r-s|}^{|r+s|} e^{-\lambda w}\,\frac{dw}{w}\,, \qquad (3.3)$$

and

$$f(x,y) = \frac{c}{2}\,e^{-\lambda x} \int_{-\sqrt{a^2-y^2}}^{x} \phi\left(\sqrt{x_1^2+y^2}\right) \frac{e^{\lambda x_1}}{\sqrt{x_1^2+y^2}}\,dx_1\,.$$

Define $\mathrm{Re}(\lambda) \equiv \beta \equiv -\gamma$. If only real values of λ are considered, then the integral operator has a real symmetric kernel and hence only real characteristic values.

Theorem 3.1. The integral transformation L_β is positive definite for real β.

Proof. The kernel of the transformation may be taken to be

$$k(r,s;\beta) = k(r-s;\beta) - k(r+s;\beta)$$

where

$$k(q;\beta) = \begin{cases} \int_{|q|}^{2a} e^{-\beta w}\,\frac{dw}{w} & \text{for } |q| < 2a \\ 0 & \text{otherwise} \end{cases}$$

If $\beta \geq 0$, consider the fourier cosine transformation ([9], p.346)

$$f_c(\nu) = \sqrt{\frac{2}{\pi}} \int_0^{2a} \cos \nu z \, dz \int_z^{2a} e^{-\beta w}\,\frac{dw}{w}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{2a} \frac{e^{-\beta w}}{w}\,\frac{\sin \nu w}{\nu}\, dw > 0 .$$

So

$$k(r,s;\beta) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} (\cos \nu|r-s| - \cos \nu\,|r+s|)\, f_c(\nu)\, d\nu$$

$$= 2\sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin \nu r \sin \nu s \, f_c(\nu)\, d\nu$$

and

$$(\phi, L_\beta \phi) = \int_0^a \phi(r)\, dr \int_0^a \overline{\phi(s)}\, ds \sqrt{\frac{8}{\pi}} \int_0^{\infty} \sin \nu r \sin \nu s \, f_c(\nu)\, d\nu$$

Only functions ϕ which are odd and zero outside the interval $(-a,a)$ need be considered. Thus,

$$(\phi, L_\beta \phi) = \sqrt{\frac{8}{\pi}} \int_0^{\infty} f_c(\nu) \left| \int_0^a \phi(s) \sin \nu s \, ds \right|^2 d\nu > 0 .$$

If $\beta < 0$, then γ is positive and

$$k(r,s;-\gamma) = \int_{|r-s|}^{|r+s|} e^{\gamma w} \frac{dw}{w} \qquad (3.4)$$

$$= 2 \int_{|r-s|}^{|r+s|} (\cosh \gamma w - 1) \frac{dw}{w} +$$

$$+ 2 \int_{|r-s|}^{|r+s|} (1 - e^{-\gamma w}) \frac{dw}{w} + \int_{|r-s|}^{|r+s|} e^{-\gamma w} \frac{dw}{w} \qquad (3.5)$$

Denote these three terms by $k_1(r,s;-\gamma)$, $k_2(r,s;-\gamma)$, and $k_3(r,s;-\gamma)$ respectively.

The first part of the proof shows that k_3 is the kernel of a positive definite transformation. The same technique suffices for k_2 , which is then also the kernel of a positive definite transformation.

Finally

$$k_1(r,s;-\gamma) = 2 \int_{|r-s|}^{|r+s|} (\cosh \gamma w - 1) \frac{dw}{w}$$

$$= 2 \int_{|r-s|}^{|r+s|} dw \int_0^{\gamma} \sinh wz \, dz$$

$$= 2 \int_0^{\gamma} (\cosh z|r+s| - \cosh z|r-s|) \frac{dz}{z}$$

$$= 4 \int_0^{\gamma} \sinh rz \sinh zs \frac{dz}{z}$$

Therefore,

$$\int_0^a \phi(r)\,dr \int_0^a \overline{\phi(s)}\, k_1(r,s;-\gamma)\,ds = 4\int_0^\gamma \left| \int_0^a \phi(s)\sinh zs\,ds \right|^2 \frac{dz}{z} > 0$$

The integral transformation L_β is a sum of positive definite transformations and hence is positive definite for all real β. Its characteristic values $\rho_1(\beta) > \rho_2(\beta) \geq \cdots$ satisfy $\rho_i(\beta) \geq 0$ for all i and all real β. The characteristic value ρ_1 is simple and has a positive characteristic function ([7], Th. 6.3).

Theorem 3.2. For each i, $\rho_i(\beta)$ is a continuous decreasing function of β.

Proof. The addition of a bounded positive definite transformation to a bounded positive definite transformation increases every characteristic value ([5], p.239). This induces a natural partial ordering among bounded positive definite operators. In the sequel $A_1 \geq A_2$ will mean that A_1, A_2 and $A_1 - A_2$ are bounded, positive semi-definite operators.

The proof is simplified by considering two separate cases.

(i) $0 \leq \beta_1 < \beta_2$

Consider

$$k(r,s;\beta_1) - k(r,s;\beta_2) = \int_{|r-s|}^{|r+s|} (e^{-\beta_1 w} - e^{-\beta_2 w})\,\frac{dw}{w}$$

and apply the fourier cosine transform to obtain

$$k(r,s;\beta_1) - k(r,s;\beta_2) = \frac{8}{\pi} \int_0^\infty \sin \nu r \sin \nu s \, d_1(\nu) \, d\nu$$

$$d_1(\nu) = \int_0^{2a} (e^{-\beta_1 w} - e^{-\beta_2 w}) \frac{\sin \nu w}{\nu w} \, dw$$

Since $(e^{-\beta_1 w} - e^{-\beta_2 w})/w$ is monotone decreasing for $0 \le \beta_1 < \beta_2$ and $w \ge 0$, it follows that $d_1(\nu) > 0$. Thus,

$$\int_0^a \phi(r) \, dr \int_0^a \overline{\phi(s)} \, (k(r,s;\beta_1) - k(r,s;\beta_2)) \, ds$$

$$= \frac{8}{\pi} \int_0^\infty d_1(\nu) \left| \int_0^a \phi(s) \sin \nu s \, dx \right|^2 d\nu > 0$$

(ii) $-\gamma_1 < -\gamma_2 \le 0$

From (3.4)

$$k(r,s;-\gamma_1) - k(r,s;-\gamma_2) = 4 \int_{\gamma_2}^{\gamma_1} \sinh rz \sinh zs \, \frac{dz}{z} +$$

$$+ \int_{|r-s|}^{|r+s|} (e^{-\gamma_2 w} - e^{-\gamma_1 w}) \, \frac{dw}{w}$$

The first term on the right hand side is clearly positive definite. The second term is exactly like the kernel of part (i) and so this part is proved.

That each $\rho_i(\beta)$ is a continuous function of β is a consequence of the L_2 continuity of $k(r,s;\beta)$ in β ([5], p.239).

The next step is to show that $\rho_i(\beta)$ tends to ∞ as β tends to $-\infty$. First observe that

$$k_1(r,s;-\gamma) = 4\int_0^{\gamma} \sinh rz \sinh zs \frac{dz}{z}$$

$$= 4\int_0^{a} \sinh \frac{\gamma}{a} rw \sinh \frac{\gamma}{a} ws \frac{dw}{w}$$

Define

$$[T_1(\gamma)]^2\phi = \int_0^a \phi(s)\, k_1(r,s;-\gamma)\, ds \qquad \gamma > 0 \qquad (3.6)$$

where $T_1(\gamma)$ is the unique positive definite square root of the right member of (3.6) ([5], p.265). Define

$$[T_0(\gamma)]^2\phi = \int_0^a \phi(s)\, k_0(r,s;-\gamma)\, ds \qquad (3.7)$$

$$k_0(r,s;\gamma) = \frac{4}{a}\int_0^a \sinh \frac{\gamma}{a} rw \sinh \frac{\gamma}{a} ws\, dw .$$

Lemma 3.3. In the natural partial ordering of bounded positive definite operators $[T_0(\gamma)]^2 < [T_1(\gamma)]^2$.

Proof. The difference of the squares of the operators is positive, since

$$(\phi, (T_1^2 - T_0^2)\phi) = 4 \int_0^a \phi(r)\, dr \int_0^a \overline{\phi(s)}\, ds \int_0^a \left(\frac{1}{w} - \frac{1}{a}\right) \sinh \frac{\gamma}{a} rw \sinh \frac{\gamma}{a} ws\, dw$$

$$= 4 \int_0^a \left(\frac{1}{w} - \frac{1}{a}\right) \left| \int_0^a \phi(s) \sinh \frac{\gamma}{a} ws\, ds \right|^2 dw > 0$$

for $\gamma > 0$.

By inspection, $T_0(\gamma)$ may be written

$$T_0(\gamma)\, \phi = \frac{2}{\sqrt{a}} \int_0^a \phi(s) \sinh \frac{\gamma}{a} rs\, ds \qquad \text{for } \gamma > 0 . \quad (3.8)$$

Lemma 3.4. The transformation $T_0(\gamma)$ has the following properties:

(i) It is the unique, bounded, symmetric, positive definite square root of $[T_0(\gamma)]^2$.

(ii) It is a compact transformation of $L_2(0,a)$ into itself.

(iii) It is a non-degenerate transformation.

Proof. The transformation is clearly bounded and symmetric. The kernel has a uniformly convergent Taylor's expansion

$$\frac{2}{\sqrt{a}} \sinh \frac{\gamma}{a} rs = \frac{2}{\sqrt{a}} \sum_0^\infty \frac{1}{(2n+1)!} \left(\frac{\gamma}{a} rs\right)^{2n+1} \qquad (3.9)$$

which is of the form $\sum k_n(r)\, \overline{k_n(s)}$. Therefore, the transformation is positive. Direct computation shows that (3.8) is a square root of T_0^2 and uniqueness is immediate ([5], p.265).

Compactness follows directly from the boundedness of the kernel ([6], p. 326).

If T_0 were of finite rank, then there would be a non-zero odd polynomial $p(s)$ such that

$$\int_0^a p(s) \sinh \frac{\gamma}{a} rs \, ds = 0$$

However,

$$\int_0^a p(s) \sinh \frac{\gamma}{a} rs \, ds = -\frac{1}{2} \int_{-a}^a p(s) \, e^{\frac{-\gamma}{a} rs} \, ds =$$

$$= -\frac{e^{\gamma r}}{2} \int_0^{2a} p(t-a) \, e^{\frac{-\gamma}{a} rt} \, dt$$

This last expression is a Laplace transform. By the uniqueness theorem for the Laplace transformation ([12], p. 35) the expression can vanish only if $p = 0$ almost everywhere in the interval of integration. Therefore T_0 is not degenerate.

All the characteristic values of T_0 have index one, because T_0 is symmetric.

Attention is now directed at the behavior of the characteristic values $\varepsilon_i(\gamma)$ of $T_0(\gamma)$ for $\gamma > 0$.

Lemma 3.5. For each $i = 1,2,\ldots$ the characteristic value $\varepsilon_i(\gamma)$ is a positive, monotone increasing function of γ which tends to ∞ as γ tends to ∞.

Proof. That $\varepsilon_i(\gamma) > 0$ is shown in the preceding lemma.

Consider $T_0(k\gamma) - kT_0(\gamma)$. This difference has the kernel

$$\frac{2}{\sqrt{a}} \left(\sinh \frac{k\gamma}{a} rs - k \sinh \frac{\gamma}{a} rs\right) \qquad (3.10)$$

$$= \frac{2k}{\sqrt{a}} \sum_{0}^{\infty} \frac{(k^{2n}-1)}{(2n+1)!} \left(\frac{\gamma rs}{a}\right)^{2n+1}$$

As for the original kernel, this function has a uniformly convergent Taylor's series in powers of rs with positive coefficients. Since this is also the kernel of a positive definite integral transformation, for $k > 1$

$$\varepsilon_i(k\gamma) - k\varepsilon_i(\gamma) > 0 \qquad (3.11)$$

Then for $\gamma > \gamma_0 > 0$ it follows that

$$\varepsilon_i(\gamma) > \varepsilon_i(\gamma_0) \frac{\gamma}{\gamma_0} > 0 \qquad (3.12)$$

Therefore $\varepsilon_i(\gamma)$ tends to infinity as γ tends to infinity.

Theorem 3.6. The characteristic values $\rho_i(\beta)$ of L_β tend to $+\infty$ as β tends to $-\infty$. Indeed, for $\beta < \beta_0 < 0$,

$$\rho_i(\beta) > \left[\varepsilon_i(\gamma_0) \frac{\beta}{\beta_0}\right]^2 .$$

Proof. By the preceding lemma, for some fixed positive $\gamma_0 = -\beta_0$, the characteristic values of $T_0(\gamma)$ satisfy

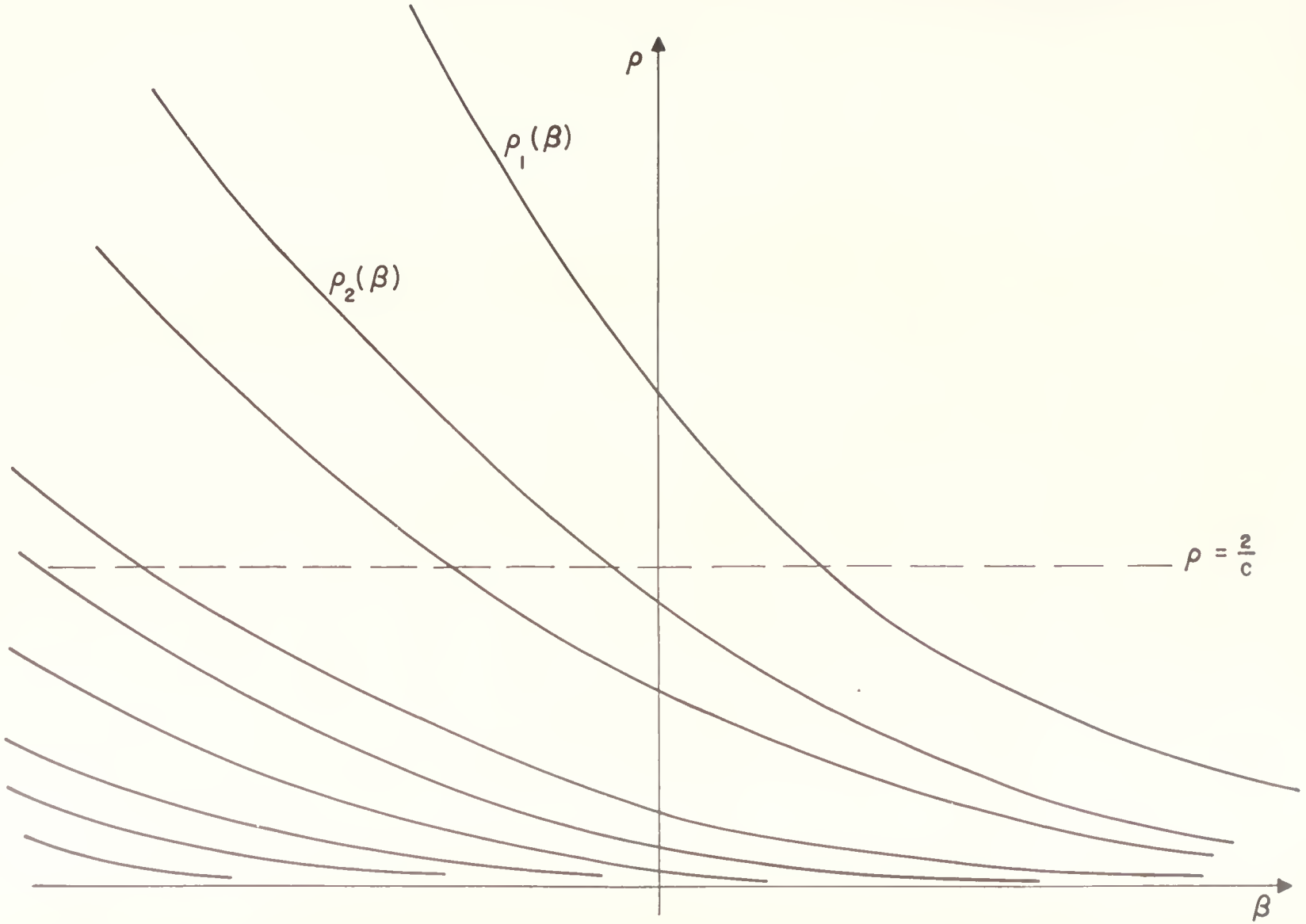

Figure 3

$$\varepsilon_i(\gamma) > \varepsilon_i(\gamma_0)\,\frac{\gamma}{\gamma_0} \qquad \text{for } \gamma > \gamma_0 .$$

The characteristic values of $[T_0(\gamma)]^2$ are $[\varepsilon_i(\gamma)]^2$. Since $L_\beta > [T_1(\gamma)]^2 > [T_0(\gamma)]^2$ for $\beta < 0$, the desired inequality is valid and the theorem is proved.

The behavior of the characteristic values of $\rho_i(\beta)$ is summarized in Figure 3. The graph may be somewhat idealized. For example, it may be possible that some curves intersect or are not always convex.

A brief recapitulation of the properties of L_β and their implications for the transport operator follows.

(i) For every fixed β the operator has a countable infinity of bounded, positive characteristic values which accumulate only at zero.

(ii) Every $\rho_i(\beta)$ is of finite multiplicity, and of index one. In particular $\rho_1(\beta)$ is simple and corresponds to a function which may be taken to be positive except at the origin.

(iii) Every $\rho_i(\beta)$ is a monotone decreasing, continuous function of β and

$$\lim_{\beta \to \infty} \rho_i = 0 \qquad \text{and} \qquad \lim_{\beta \to -\infty} \rho_i = \infty$$

Therefore for every $\rho_i(\beta)$ there is a unique β_i such that $\rho_i(\beta_i) = \frac{2}{c} > 0$.

By using the second relation of (3.3) a characteristic function of A, the transport operator, may be found whose characteristic value is $\beta_i - \sigma$.

(iv) Therefore the transport operator has a countable infinity of real characteristic values accumulating at minus infinity. Each of these characteristic values corresponds to a finite number of characteristic functions.

Finally, it will be shown that the real characteristic values β_i of the transport operator have index one.

Lemma 3.7. If $Af = \beta f$ and $f^*(x,y) = \overline{f(-x,y)}$, then $(f,f^*) \neq 0$.

Proof. A lengthy computation is necessary.

$$\frac{1}{4\pi}(f,f^*) = \int_0^a y\,dy \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x,y)\, f(-x,y)\, dx$$

$$= \frac{c^2}{4}\int_0^a y\,dy \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dx \int_{-\sqrt{a^2-y^2}}^{x} dp \int_x^{\sqrt{a^2-y^2}} dq\; e^{-\lambda(q-p)}\, \frac{\phi(\sqrt{p^2+y^2})}{\sqrt{p^2+y^2}}\, \frac{\phi(\sqrt{q^2+y^2})}{\sqrt{q^2+y^2}}$$

$$= \frac{c^2}{4}\int_0^a dw \int_{-w}^{w} dx \int_{-w}^{x} dp \int_x^w dq\; w\, e^{-\lambda(q-p)}\, \frac{\phi(\sqrt{p^2+a^2-w^2})}{\sqrt{p^2+a^2-w^2}}\, \frac{\phi(\sqrt{q^2+a^2-w^2})}{\sqrt{q^2+a^2-w^2}}$$

$$= \frac{c^2}{8}\int_0^a dw \int_{-w}^{w} dq \int_{-w}^{w} dp\; |p-q|\, w\, e^{-\lambda|p-q|}\, \frac{\phi(\sqrt{p^2+a^2-w^2})}{\sqrt{p^2+a^2-w^2}}\, \frac{\phi(\sqrt{q^2+a^2-w^2})}{\sqrt{q^2+a^2-w^2}}$$

$$= \frac{c^2}{2}\int_0^a dw \int_0^w dq \int_0^q dp\; (|p-q|\, e^{-\lambda|p-q|} + |p+q|\, e^{-\lambda|p+q|})\, w \ldots$$

$$= \frac{c^2}{2}\int_0^a dr \int_r^a ds \int_{\sqrt{a^2-r^2}}^{a} dw\; \phi(r)\,\phi(s)\, w\Big(\Big|\frac{1}{\xi(r)} - \frac{1}{\xi(s)}\Big| e^{-\lambda|\xi(r)-\xi(s)|} +$$

$$+ \Big|\frac{1}{\xi(r)} + \frac{1}{\xi(s)}\Big| e^{-\lambda|\xi(r)+\xi(s)|}\Big)$$

where

$$\xi(r) = \sqrt{r^2-a^2+w^2}$$

then

$$\frac{1}{4\pi}(f,f^*) = \frac{c^2}{2\lambda}\int_0^a dr \int_r^a ds\ \phi(r)\ \phi(s)\ (e^{-\lambda|r-s|}-e^{-\lambda|r+s|}) \qquad (3.13).$$

$$= \frac{c^2}{4\lambda}\int_0^a dr \int_0^a ds\ \phi(r)\ \phi(s)\ (e^{-\lambda|r-s|}-e^{-\lambda|r+s|})$$

In the case where $\lambda = \beta$ is real, then the characteristic function ϕ may be taken to be real.

$$\frac{1}{4\pi}(f,f^*) = \frac{c^2}{4\beta}\int_0^a dr \int_0^a ds\ \phi(r)\ \phi(s)\ (e^{-\beta|r-s|} - e^{-\beta|r+s|})$$

If $\beta \geq 0$, apply the fourier cosine transformation to the kernel of the quadratic form (3.13). This kernel is to be modified exactly as in Theorem 3.1.

$$\frac{1}{\beta}(e^{-\beta|r-s|} - e^{-\beta|r+s|}) = \int_{|r-s|}^{|r+s|} e^{-\beta w}\ dw$$

$$= \frac{4}{\pi}\int_0^\infty d\nu \sin \nu r \sin \nu s \int_0^{2a} dp \cos \nu p \int_p^{2a} e^{-\beta w}\ dw$$

$$= \frac{4}{\pi}\int_0^\infty d\nu \sin \nu r \sin \nu s \int_0^{2a} dw\ \frac{e^{-\beta w}}{\nu} \sin \nu w\ .$$

The monotonicity of the exponential makes the inner integral

positive except for $\beta = 0$ and $a\nu = k\pi$, when it is zero. Then

$$(f, f^*) = 4c^2 \int_0^\infty \frac{d\nu}{\nu} \left(\int_0^a ds\ \phi(s) \sin \nu s \right)^2 \int_0^{2a} dw\ e^{-\beta w} \sin \nu w$$

$$> 0 \qquad \text{for } \beta \geq 0 .$$

If $\beta < 0$, then let $\beta = -\gamma$ and the kernel of the quadratic form is

$$\frac{1}{\gamma} \left(e^{\gamma|r+s|} - e^{\gamma|r-s|} \right) = \frac{4}{\gamma} \sinh \gamma r \sinh \gamma s + \frac{1}{\gamma} \left(e^{-\gamma|r-s|} - e^{-\gamma|r+s|} \right)$$

In this case

$$(f, f^*) = \frac{4\pi c^2}{\gamma} \left(\int_0^a ds\ \phi(s) \sinh \gamma s \right)^2 +$$

$$+4c^2 \int_0^\infty \frac{d\nu}{\nu} \left(\int_0^a ds\ \phi(s) \sin \nu s \right)^2 \int_0^{2a} dw\ e^{-\gamma w} \sin \nu w > 0$$

$$\text{for } \gamma \geq 0 .$$

Thus, for each β_i the characteristic functions of the operator and its adjoint can be taken to be biorthogonal ([10], p.266).

Theorem 3.8. The index of the real eigenvalues of A is one.

Proof. If the index of β is greater than one, then there is an f such that

$$(\beta - A)^2 f = 0 \quad \text{and} \quad (\beta - A) f \neq 0$$

([11], Th. 10.8). Let β have multiplicity s and

$$(\beta - A)\, f_j = 0 \quad \text{for} \quad j = 1, 2, \ldots, \quad s \geq 1 \ ;$$

then also

$$(\beta - A^*)\, f_j^* = 0 \quad \text{for} \quad j = 1, 2, \ldots, \quad s \geq 1$$

where $f_j^*(x, y)$ is a linear combination of the $\overline{f_j(-x, y)}$. Since $(\beta - A)\, f$ is a characteristic function of A,

$$(\beta - A)\, f = \sum_{j=1}^{s} a_j\, f_j$$

If the sets $[f_j]$ and $[f_j^*]$ are biorthogonal, then

$$a_j = ((\beta - A) f, f_j^*) = (f, (\beta - A^*) f_j^*) = 0 \qquad j = 1, 2, \ldots, s$$

and $(\beta - A)\, f = 0$ contrary to hypothesis.

4. Remarks

At this writing it has been impossible to prove the existence or non-existence of complex characteristic values for $Re(\lambda) < 0$. That no naive extension of the method used by Lehner and Wing can be used is shown as follows. Let

$$L_\lambda \equiv M_\lambda + iN_\lambda$$

where M_λ and N_λ are real symmetric operators. If for real, positive ρ and for real functions ϕ and ψ

$$L_\lambda(\phi+i\psi) = \rho(\phi+i\psi) ,$$

then

$$\rho\phi = M_\lambda\phi - N_\lambda\psi$$

and

$$\rho\psi = N_\lambda\phi + M_\lambda\psi .$$

Taking inner products and subtracting yields

$$(\phi, N_\lambda\phi) + (\psi, N_\lambda\psi) \equiv Im((\phi + i\psi) , L_\lambda(\phi + i\psi)) = 0 .$$

Therefore, if N_λ is definite (as it is for $Re(\lambda) \geq 0$, $Im(\lambda) \neq 0$), there can be no real, positive characteristic

value ρ . This is the essence of the proof given by Lehner and Wing. In general, however, $(\phi+i\psi, L_\lambda(\phi+i\psi))$ is

(i) an entire function of λ

(ii) not a polynomial

(iii) real and monotone decreasing for λ real.

The function must assume every real, positive value (with the possible exception of a single value) infinitely often. Since it can assume a real value ρ at most once on the real line and not at all for the right half plane off the real line, it must take this value in the left half plane.

It is possible to exclude from the spectrum a band about the negative real line (and excluding that) whose width decreases exponentially as $Re(\lambda) \longrightarrow -\infty$. By use of comparison kernels it can be shown that the imaginary part N_λ of the transformation is definite in this band.

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